Hedge Fund Portfolio Management with Illiquid Assets

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Abstract

We study hedge fund optimal portfolios in the presence of market and funding liquidity risks. We consider a two-period economy with a single hedge fund. The fund has access to cash which is available every period and to an illiquid asset which pays off only at the end of the second period. Funding liquidity risk takes the form of a random proportion of the fund's assets under management being withdrawn by clients in period one. The fund can then liquidate a part of the illiquid position by bidding on a secondary market where a random haircut on the effective selling price is applied. We solve the allocation problem of the fund and find its optimal portfolio. Whereas the cash buffer is monotonously decreasing in the secondary market liquidity, we show that the fund's default probability is bell-shaped. Finally, we apply our model in an asset pricing framework for different hedge fund strategies to see how both risks are priced over time.

JEL Codes: G11, G12.

Key-words: Hedge Fund, Liquidity, Default.

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1 Introduction

Hedge Funds are particularly exposed to liquidity risk. First, they can easily invest in illiquid securities or long-term projects which expose them to market liquidity shocks. Second, hedge funds can offer their investors the opportunity to withdraw their capital on short horizons, and are exposed to funding liquidity issues. This has driven the academic literature to focus on the impact of both liquidity risks on the funds' performance. However, the bulk of existing studies are empirical works considering either market or funding liquidity risk, but not both. This separation seems to be at odds with management practice: funds are likely to design their portfolios with respect to their potential liquidity mismatch between their asset and their liability sides. For example, a quarterly liquidity fund investing in illiquid assets can be less risky than a daily liquidity fund investing in liquid assets because of its natural hedge against short term funding liquidity shock. In addition, funds can continuously adjust their exposures to liquidity shocks depending on current funding and market conditions, thus raising an endogeneity issue.

In this paper, we formulate a new structural model to study the optimal behavior of a hedge fund when its investments are exposed to both funding and market liquidity risks. This framework allows the fund to adjust its optimal liquidity cushion to not only the observed market conditions but also to the risks underlying its own liability side. As such, we are able to create endogenous default events, which arise from the impossibility for the fund manager to reimburse its clients' withdraws. It corresponds to the conjunction of a high negative shock on its liability side (funding liquidity shock) and/or poor market conditions to liquidate the risky assets on its balance sheet (market liquidity shock). Our model hence provides a first attempt to solve the endogeneity issue present in most empirical studies on hedge fund liquidity risk exposures.

Our modeling approach lies as follows. We formulate a two-period model where a single hedge fund entity has a stylized balance-sheet composed of assets under management on the liability side, and both cash and an illiquid investment on the asset side. Cash is assumed to pay off a zero interest rate at every period, whereas the illiquid asset pays off a positive interest rate (known *ex-ante*) only at the end of the second period. There are two shocks in the model. The *funding liquidity shock* is represented by a random proportion of assets under management being withdrawn from the hedge fund at the end of the first period. To cover its losses, the fund can use its available cash. If its cash quantity is not sufficient to reimburse its clients, the fund can sell its illiquid asset on a secondary market. The *market liquidity shock* takes the form of a random haircut being applied to the theoretical friction-less return of the illiquid asset on the secondary market. The fund defaults whenever the effective selling price is such that the fund cannot cover the capital outflow.

We model the funding liquidity shock as a uniformly-distributed variable on a subsegment of $[0, 1]$. On the other hand, the market liquidity shock (i.e. the rebate on the illiquid asset's return) is specified as an exponentially-distributed variable. The model has five parameters in total: the probability of having a positive funding liquidity shock, the maximum possible funding liquidity shock, the average rebate on the secondary market, and the friction-less returns of the illiquid asset in periods one and two. The hedge fund chooses its optimal amount of cash to maximize its compounded expected returns over the two periods with respect to these parameters. A particular advantage of this framework is that, for any set of parameters and a given cash amount, both the expected value of the fund's portfolio and the fund's default probability are computable in closed-form.

Even though the two shocks are independent, the optimal cash amount and the optimal default probability are functions of both funding and market liquidity conditions. In particular, when the funding liquidity shock can be as high as 100% of the liability side, we show that both these optimal values are available in closed-form. The model can be easily numerically solved otherwise. The availability of a closed-form solution allows us to derive analytical results on the model.

Three testable hypotheses emerge from our model. First, stricter withdrawal restrictions generate higher expected portfolio value at the optimum. These funds happen to have adequate capital to fund and maintain their trading strategies. Second, the optimal cash holding is decreasing in the secondary market liquidity, but the fund's default probability is a bell-shaped function of the market liquidity. Finally, when the fund can experience 100% withdrawals, its optimal cash holdings are such that default probability is insensitive to the probability of a funding liquidity event.

In a second part, we apply our theoretical framework on observed data for different fund management styles to test the model predictions. The data contains actual contractual provisions, observed performance, flows and default events observed monthly from the 2000s on, which allows us to calibrate the model's parameters. For each period, we use natural estimators to compute the probability of being exposed to positive outflows and their maximum possible amounts, the average return of the illiquid assets, and the default probabilities within a management style. Time series for each parameters are obtained by minimizing a cost function between model-implied and empirical quantities in a method-of-moment-type estimation. For each management style, we provide the series of the liquidity cushion held by the representative hedge fund, the average rebate on the secondary market, and a liquidity pricing factor computed as the difference between the illiquid asset return and the fund's optimal expected return. Using these series, we show that even though they exhibit imperfect correlation with existing risk factors, our liquidity factors are significantly priced in the hedge fund's performance.

The consequence of market and funding liquidity shocks on Hedge Fund performance has been intensively studied in the academic literature. However, most of these studies consider separately the two liquidity risks, and only a few papers focus on the consequence of both market and funding liquidity risks on hedge fund performance and survival.

Sadka (2010, 2012) examines funds' exposure to aggregate market-wide liquidity. He investigates whether liquidity risk is priced in hedge fund returns using different liquidity risk factors and finds that funds with high exposure to aggregate liquidity risk outperform those with low exposure by 6% annually during normal months. However, during periods where liquidity is scarce, these funds with high liquidity risk drastically underperform those with low exposure. Teo (2011) studies the performance of the most liquid hedge funds, i.e., those that offer the shortest lockup and redemption notice periods to their investors. He finds that liquid funds that experience net capital outflows may be forced into fire sales, and suffer lower risk-adjusted returns (by 4.79 percent) than their counterparts with high net inflows. Finally, Jame (2015) uses transaction-level data to study whether hedge funds profit from providing liquidity to the market. He focuses in particular on liquidity provision as a source of hedge fund performance.

Dudley and Nimalendran (2010) show that the funding liquidity risk is priced in hedge fund returns. They first develop a funding liquidity risk factor for hedge funds using the residuals of a regression of futures margins on the implied volatility index (VIX). They then include their factor in classic asset pricing tests such as portfolio sorts and factor model regressions. Liu and Mello (2011) relate funding liquidity risk to funds' cash holdings. The optimal cash holding reflects a trade-off between the reduction of liquidation costs and the increase in returns by holding risky assets. They find that investors fear that other investors may withdraw their capital and force a fire sale. They suggest that redemption risk led hedge funds to hold more cash to resist to funding liquidity shocks. Finally, Aragon and Strahan (2012) use the Lehman Brothers bankruptcy as an exogenous shock to hedge funds' funding liquidity. Lehman Brothers was the prime broker for many hedge funds in 2008. The authors find that the Lehman bankruptcy in September 2008 increased by 50% the default probability of funds using this prime broker during the next year. However, they also stress that it is difficult to empirically measure the impact of funding liquidity risk on hedge funds since their strategies and funding arrangements are jointly chosen. To circumvent this empirical issue, Hombert and Thesmar (2014) develop a model *à la* Shleifer and Vishny (1997). They assume that managers choose a set of contractual features that impact how sensitive capital is to poor performance. Therefore, lockup and notice periods restrict investors' ability to withdraw their capital after a period of poor performance. The authors test the model predictions using selfreported liquidity restrictions, estimated flow-performance sensitivity and a new measure that captures the ability of lockup provisions to retain capital inflows. They find that fund managers effectively choose contractual features to reduce investors' ability to withdraw their capital.

Considering simultaneously fund and market liquidity risks, Agarwal, Aragon, and Shi (2015) focus on funds of hedge funds and propose a measure of the difference (named liquidity gap) between the liquidity of fund of funds' assets (i.e., underlying hedge funds in the fund of funds portfolio) and the liquidity of fund of funds liabilities (i.e., redemption by investors). Funds of funds with larger illiquidity gaps are shown to perform badly during crises and exhibit greater exposure to runs (i.e. some investors redeeming strategically prior to others when funds perform poorly). As in Jame (2015), Franzoni and Plazzi (2013) study whether hedge funds profit from providing liquidity to the market. They find that the level of liquidity provision decreases when funding and market conditions deteriorate. They use the VIX index, the TED spread, and the LIBOR rate as proxies for trading costs since these measures are related to the costs of borrowing or to the tightness of margin requirements. Hedge funds' ability and willingness to provide liquidity during times of lower market liquidity is also related to their own fund flows. Finally, a decline in hedge fund trading predicts a decline in liquidity at the individual stock level.

Our contributions are threefold. First, as in Agarwal, Aragon, and Shi (2015), we consider jointly market and funding liquidity risks, but our model endogenizes the HF manager behavior in terms of liquidity risk hedging. Second, we obtain the fund default probability and the liquidity management cost as a function of the market and funding liquidity conditions. We explain how fund managers adjust cash holdings to immunize their fund against funding liquidity shocks. We confirm the first results obtained by Hombert and Thesmar (2014) on contractual provisions management and explain why default probabilities are not increasing too much during the financial crisis (compared to banks' default probabilities). Third, we build a new liquidity factor that reflect the implied cost of the liquidity mismatch between assets and liabilities. This factor is depending on both the market and the funding liquidity conditions, and not only market [see e.g. Sadka (2010, 2012)] or funding [see e.g. Fontaine and Garcia (2012)] liquidity conditions.

Our model can be used by three kinds of market participants. Hedge Fund managers need to calibrate the optimal level of cash holdings depending market and funding liquidity risks. Investors need to understand the sources of alpha and, in particular, the (nonlinear) relationship between alpha and liquidity risks. Finally, for regulators, our model stresses the importance to track outflows in the computation of liquidity risk.

The paper is organized as follows. Section 2 describes the setup used to endogeneizes the HF manager behavior. Section 3 describes how the model is solved to get the optimal csh holding level and lists the model implications. Section 4 uses hedge fund data to test the model predictions and find evidence supportive of each. Section 5. All the proofs are gathered in the appendices.

2 The model

This section provides a description of the setup used to represent the fund manager behavior when both market and funding liquidity shocks can occur. Section 2.1 introduces the main assumptions, while Section 2.2 details the computation of the conditional fund returns given a chosen quantity of cash. Section 2.3 provide closed form solution for the fund's default probability and expected returns for a given level of cash holdings.

2.1 Setup

We consider a two-period economy with a single Hedge Fund (HF henceforth) entity which is endowed with one unit of capital before the beginning of the first period ($t = 0$).[2] This HF has access to two assets. The first one is a short-term risk-less asset which pays off a zero interest rate every period. We equivalently refer to this asset as "cash" or "liquid asset" hereafter. The second one is an asset which pays off only at the end of the second period ($t = 2$), hence embeds positive liquidity risk. We denote by $(\rho_1 + \rho_2)$ the log-return of this illiquid asset for the two periods. In a friction-less world, the HF could liquidate its illiquid positions such that it earns a gross rate $\exp(\rho_1)$ on the first period and $\exp(\rho_2)$ between the first and the second period. We assume that $\exp(\rho_1)$ and $\exp(\rho_2)$ are known ex-ante and greater than one, compensating for the underlying liquidity risk.

We assume that, at the end of period one ($t = 1$), a *funding liquidity* shock can occur with probability π. This liquidity shock takes the form of a proportion of assets under management being withdrawn from the fund's liability side. This quantity is modeled as a random uniform variable $\theta \sim \mathcal{U}(0, \overline{\theta})$. The upper limit $\overline{\theta} \in (0, 1]$ represents any legal or credible limit to the maximum outflows in the hedge fund. The problem of the HF manager is to find the optimal portfolio $(\delta, 1 - \delta)$ at the beginning of period one ($t = 0$) provided it can suffer from a liquidity shock at the end of this period ($t = 1$). Would it not be able to reimburse its investors after a liquidity shock, the fund defaults. Without additional assumptions, an obvious solution would be for the fund to invest $\delta = \overline{\theta}$ in the liquid asset such that it avoids failing in any case.

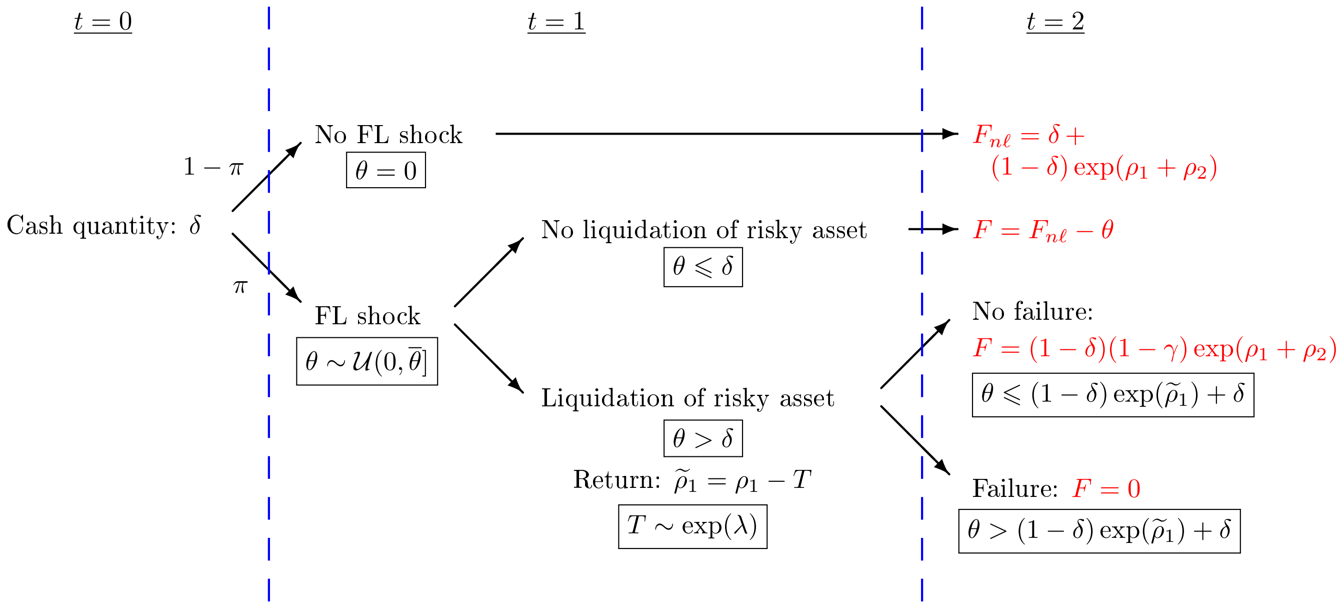
[2] Our model in its current form does not consider leverage, i.e. the possibility to invest more than the clients' capital.

We add another layer to this simplistic setup considering that the HF can liquidate his position on the illiquid asset by bidding on a secondary market to avoid bankruptcy. Doing so, it suffers from a random haircut on the effective selling price. Let P_1 be the (virtual) friction-less price of the risky asset at period $t = 1$, the observed selling price $\widetilde{P}_1$ on the secondary market is given by:

$$\widetilde{P}_1 = P_1 \exp(-T) \quad \text{where} \quad T \sim \mathcal{E}(\lambda)\,. \tag{1}$$

The random variable T can be interpreted as a stopping time: when the market liquidity is low, it is likely that the fund will wait for more time before finding a counterparty. It would thus have to bid with lower prices to avoid bankruptcy. The value $1/\lambda$ is the average waiting time before liquidating the asset. Hence, when λ is high, the secondary market is very liquid and λ represents the degree of *market liquidity*. Letting $\widetilde{\rho}_1$ be the effective (realized) log rate of return on the secondary market, we trivially have $\widetilde{\rho}_1 = \rho_1 - T$, and the gross rate of return on the secondary market is given by $\exp(\widetilde{\rho}_1) = \exp(\rho_1 - T)$. The fund defaults whenever $\widetilde{\rho}_1$ is too low to cover the gap between the funding liquidity shock θ and the quantity of cash δ, that is $\theta > (1-\delta)\exp(\widetilde{\rho}_1) + \delta$. Figure 1 summarizes the setup.

Figure 1: Summary of the setup



Notes: "FL" stands for "funding liquidity". γ is the quantity of the risky asset that needs to be liquidated to cover the funding liquidity shock θ.

2.2 Liquidity scenarois

We now detail the fund returns with respect to the three possible scenarios where the gross returns are positive, as described in Figure 1. We assume that the fund has chosen the proportion δ of the liquid asset. In the first scenario, the fund is not subject to a funding liquidity shock. The HF manager does not reshuffle its portfolio and obtains a final value of:

$$F_{n\ell} = \delta + (1-\delta)\exp(\rho_1 + \rho_2)\,. \tag{2}$$

In the second scenario, a "low" funding liquidity shock happens, small enough such that the HF does not need to liquidate its risky asset. This situation appears if and only if the quantity of risk-less asset is sufficient to cover the liability outflow, i.e. $\theta \leqslant \delta$. In this scenario, the fund's portfolio value at the end of the second period is given by:

$$F_{\ell_s} \;=\; (\delta-\theta) + (1-\delta)\exp(\rho_1+\rho_2) = F_{n\ell} - \theta. \tag{3}$$

In the third scenario, the fund is subject to a big funding liquidity shock, and $\theta > \delta$. In this situation, the HF manager has to liquidate some proportion of the risky asset on the secondary market. He thus needs to find an amount of money corresponding to $\theta - \delta$. He sells an amount of risky asset $\gamma_{\theta,T}$ such that it can exactly cover this discrepancy:

$$\gamma_{\theta,T} = \frac{\theta-\delta}{(1-\delta)}\exp(T-\rho_1)\,. \tag{4}$$

This quantity gathers two different sources of randomness, namely the size of the funding liquidity shock θ and the size of the market liquidity shock T. The fund's portfolio in this scenario is equal to:

$$F_{\ell_b} = \begin{cases} (1-\gamma_{\theta,T})(1-\delta)\exp(\rho_1+\rho_2) & \text{if} \quad \dfrac{\theta-\delta}{(1-\delta)}\exp(T-\rho_1) \leqslant 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Indeed, this return is only realized if the quantity of risky asset that can be liquidated is big enough to cover the missing liability, that is if $\gamma_{\theta,T} \leqslant 1$.

Denoting by F the fund's portfolio value, we express its expectation from period $t=0$ using the law of iterated expectations:

$$\begin{aligned} \mathbb{E}(F) \;=\;& (1-\pi)\,\mathbb{E}(F_{n\ell}) + \pi\mathbb{P}(\theta\in(0,\delta])\,\mathbb{E}(F_{\ell_s}|\theta\in(0,\delta]) \\ &+\; \pi\,\mathbb{P}(\theta>\delta)\,\mathbb{P}(\gamma_{\theta,T}\leqslant 1|\theta>\delta)\,\mathbb{E}(F_{\ell_b}|\theta>\delta,\gamma_{\theta,T}\leqslant 1) \end{aligned} \tag{5}$$

From the second row of Equation (5), we see the survival probability appearing as $\pi\,\mathbb{P}(\theta > \delta)\,\mathbb{P}(\gamma_{\theta,T} \leqslant 1|\theta > \delta)$. This probability is the joint probability of experiencing a funding liquidity shock which is big enough to trigger liquidation of the risky asset, and that the haircut on the secondary market is small enough to be alive at the last period.

2.3 Default probability and expected fund's portfolio value

From the above framework, it is clear that both funding and market liquidity risk are going to intervene in the computation of the probability of failure of the HF. Indeed, for a chosen quantity of cash δ, the market liquidity –as represented by λ– matters more when the funding liquidity shock θ is high. We first obtain the following lemma.

Lemma 2.1 *Given a cash holding level δ and the occurrence of a big liquidity shock ($\theta > \delta$), the conditional default probability is given by:*

$$\mathbb{P}\left(\gamma_{\theta,T} > 1|\theta > \delta\right) = \frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}. \tag{6}$$

When $\overline{\theta} = 1$, the conditional default probability is independent from δ and is given by:

$$p_c(\lambda,\rho_1) := \frac{\exp(-\lambda\rho_1)}{\lambda+1}.$$

Proof See Appendix A.1. ■

This first result will be used to simplify Equation (5). Moreover, it can be used to easily obtain the marginal default probability as a function of the cash holding level.

Proposition 2.1 *Given a cash holding level δ, using Lemma 2.1, the HF default probability at period $t = 2$ is given by:*

$$\mathbb{P}(\gamma_{\theta,T} > 1) = \frac{\pi\left(\overline{\theta}-\delta\right)^{\lambda+1}}{\overline{\theta}(\lambda+1)(1-\delta)^{\lambda}}\exp(-\lambda\rho_1). \tag{7}$$

Proof See Appendix A.1. ■

Note that the quantity of cash δ chosen by the HF is always below $\overline{\theta}$. Hence the default probability is always positive and is exactly equal to zero when $\delta = \overline{\theta}$. We can then study the HF default probability depending on the market and funding liquiditty conditions.

Corollary 2.1.1 *Given a cash holding level δ, The HF default probability at period $t = 2$ is:*

i. decreasing in the liquidity of the secondary market λ;

ii. increasing in the maximum size of the funding liquidity shock $\overline{\theta}$;

iii. decreasing in the quantity of cash δ.

Proof See Appendix A.2. ∎

When the secondary market is totally illiquid ($\lambda = 0$), the default probability is equal to the joint probability of a liquidity shock and that the size of the shock is bigger than the quantity of cash, that is $\pi(1 - \delta/\overline{\theta})$. When the secondary market is perfectly liquid ($\lambda \to +\infty$), the default probability converges to zero.

We are now able to derive the expected fund's portfolio value in a closed-form. Using Proposition 2.1, we obtain the following expression of the expected returns of Equation (5).

Proposition 2.2 *The expected fund's value is given by:*[3]

$$\begin{aligned} \mathbb{E}(F) &= (1-\pi)\left[\delta + (1-\delta)\exp(\rho_1+\rho_2)\right] + \pi\frac{\delta}{\overline{\theta}}\left((1-\delta)\exp(\rho_1+\rho_2) + \frac{\delta}{2}\right) \\ &+ (1-\delta)\exp(\rho_1+\rho_2)\pi\frac{\overline{\theta}-\delta}{\overline{\theta}}\left[1 - \frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\right] \\ &\times \left\{1 - \frac{\lambda}{1-\lambda^2}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\left[\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda-1} - \frac{\lambda+1}{2}\right]\right\}. \end{aligned} \tag{8}$$

Proof See Appendix A.3. ∎

On the second row of Equation (8) we have the probability of having a big funding liquidity shock and to survive to this shock (see Equation (7)). Obviously, the probability of having a big funding liquidity shock is increasing in $\overline{\theta}$ whereas the survival probability is decreasing in $\overline{\theta}$. It is also easy to see that the conditional expected fund's value in this situation is decreasing in $\overline{\theta}$. This yields an overall ambiguous effect of $\overline{\theta}$ on the expected value, thus on the optimal cash amount chosen by the HF. Note that the size of this effect also depends non-linearly on the gross return on the risky asset $\exp(\rho_1)$ and on the liquidity of the market λ.

[3] Even though the function is ill-defined for $\lambda = 1$, it is possible to calculate it in closed-form as presented in Appendix A.3. It can be easily shown the the fund's value for $\lambda \neq 1$ converges to the function presented in the appendix when λ approaches 1. A continuity argument can be applied without loss of generality.

3 Model solution and implications

In this section, we relax the assumption that the HF cash holdings are given. We consider in Section 3.1 that the HF manager optimizes its cash holdings at time $t = 0$. We then list in Section 3.2 a series of model implications in the special case, and then extend the analysis to the general case in Section 3.3.

3.1 Model solution when $\overline{\theta} = 1$

The program of the HF manager is o maximize the expected fund's portfolio value under no short-selling. We have:

$$\delta^* = \underset{\delta}{\operatorname{argmax}}\, \mathbb{E}[F(\delta, \overline{\theta}, \lambda, \pi, \rho_1, \rho_2)] \quad \text{s.t.} \quad \delta \in [0, 1]\,.$$

In the following, we show that $\overline{\theta} = 1$ allows for an analytical derivation of most results and postpone the analysis of the influence of $\overline{\theta}$ to Section 3.3.

Proposition 3.1 *When $\overline{\theta} = 1$, the optimal amount of cash $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ is given by:*

$$\delta^*(1, \lambda, \pi, \rho_1, \rho_2) = \frac{G(\lambda, \rho_1) + H_1(\pi, \rho_1, \rho_2)}{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)} \quad \textit{if this ratio is in } (0, 1)\,, \tag{9}$$

with

$$\begin{aligned} G(\lambda, \rho_1) &= \left(1 - \frac{\exp(-\lambda\rho_1)}{\lambda + 1}\right)\left(1 + \frac{\lambda\exp(-\rho_1)}{2(1-\lambda)} - \frac{\lambda\exp(-\lambda\rho_1)}{1-\lambda^2}\right), \\ H_1(\pi, \rho_1, \rho_2) &= \left(\frac{1}{\pi} - 1\right)\frac{1 - \exp(-\rho_1 - \rho_2)}{2} - \frac{1}{2}, \\ H_2(\rho_1, \rho_2) &= \frac{\exp(-\rho_1 - \rho_2)}{2} - 1\,. \end{aligned}$$

If the ratio $\frac{G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)}$ is negative or greater than 1, then $\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$ is respectively equal to 0 and 1.

Proof See Appendix A.4. ■

The representation of Proposition 3.1 is convenient as it allows to separate the influence of each exogenous variable on the optimal quantity of cash. In the ratio defining $\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$, the influence of λ goes only through the function $G(\lambda, \rho_1)$, and π is only present in the function $H_1(\pi, \rho_1, \rho_2)$.

3.2 Direct model implications when $\overline{\theta} = 1$

Proposition 3.2 *When $\overline{\theta} = 1$, the following relationships are verified:*

i. $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ *is monotonously increasing with* π*.*

ii. $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ *is monotonously decreasing with* λ*.*

iii. The HF optimal expected value when it owns $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ *of cash are monotonously increasing with* λ*.*

Proof See Appendix A.4. ■

Proposition 3.2.*i.* is consistent with the intuition behind Proposition 3.1. If the probability that the HF loses some of its clients becomes higher, its buffer as represented by the quantity of cash $\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$ should increase to avoid bankruptcy. Corollary 3.2.*ii.* and Corollary 3.2.*iii.* arise because of a dominating substitution effect. Indeed, when the liquidity of the secondary market is higher, the relative cost of the risky asset becomes lower, increasing its desirability. By decreasing its optimal amount of cash, the HF dampens its funding liquidity to increase its expected future value. Indeed, decreasing its cash quantity implies a non-zero effect on its default probability at period 2. However, Corollary 3.2.*iii.* emphasizes that any possible effect on the default probability is dominated by the increase in the fund's value conditionally on no default.

We discuss the effects of the market liquidity on the default probability in the following proposition.

Proposition 3.3 *When* $\overline{\theta} = 1$*, the HF default probability when it holds the optimal cash quantity* $\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$ *:*

$$\mathbb{P}(\gamma_{\theta,T} > 1) = \pi \left(\frac{H_2(\rho_1, \rho_2) - H_1(\pi, \rho_1, \rho_2)}{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)} \right) p_c(\lambda, \rho_1),$$

is insensitive to variations in π*, and is a bell-shaped function of the market liquidity* λ*.*

Proof See Appendix A.5. ■

The HF therefore picks its cash quantity such that its default probability is insensitive to any variation of the total probability of having a positive funding liquidity shock. However, when the liquidity of the secondary market increases, the default probability first increases, then decreases. Up to a certain value $\overline{\lambda}$, any marginal increase in λ is a big improvement in the liquidity of the secondary market. The HF conditional default probability $p_c(\lambda, \rho_1)$ which is decreasing convex in λ thus drops, making the illiquid asset more profitable relatively to holding cash. Therefore, the substitution effect dominates when there is an increase in the market liquidity. Past $\overline{\lambda}$, the effect on the conditional default probability and the expected fund's value fades out and the substitution from cash to risky asset is small enough such that the increase in liquidity dominates the effect on the total default probability.

This particular pattern can be observed on the first row of Figure 2, where we plot the optimal cash amount δ^* (left column) and the default probability when the cash is at the optimal value (right column) for $\pi = \{0.5, 0.75, 1\}$, $\overline{\theta} = 1$ and $\rho_1 = \rho_2 = 3\%$. The smaller the probability of a funding liquidity shock, the more rapidly the optimal cash amount converges to zero. In particular when $\pi = 50\%$, the fund's cash amount is equal to zero as soon as the average rebate on the secondary market is lower than 33% ($\lambda = 3$). Thus, whenever the fund expects to get more than a -30% return on the secondary market, it will load entirely on the illiquid investment (see left column, first row, Figure 2). Interestingly, this value of λ is approximately the value where the default probability begins to decrease, after reaching a highest of 12%. As given by Proposition 3.3, the default probability is invariant to π up until the "no short-selling" constraint begins to bind.

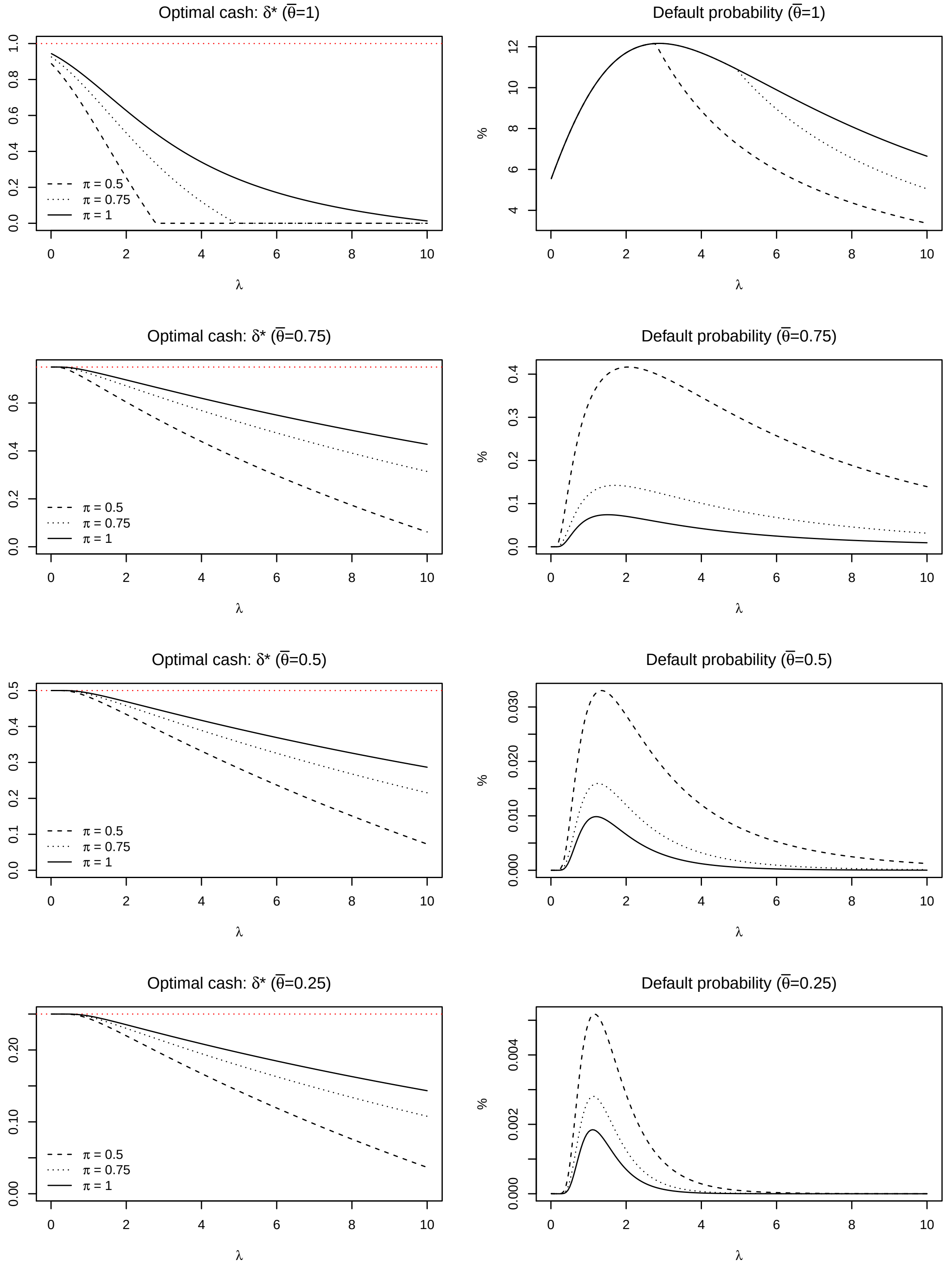
3.3 Reduced size liquidity shock: $\overline{\theta} < 1$

In this Section, we gauge the influence of the maximum funding liquidity shock parameter $\overline{\theta}$. Since the optimal cash amount is not available in closed-form when $\overline{\theta} < 1$, we use numerical techniques to obtain several examples.

Proposition 3.4 *When $\overline{\theta} < 1$, all relationships of Proposition 3.2 still hold. In addition, the default probability when the fund holds the optimal amount of cash $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ is still bell-shaped in λ but is decreasing with π.*

Though no analytical proof is available for this result, these properties can be observed on both Figures 2 and 3. Looking at the left column of Figure 2, we see that whatever the probability of a funding liquidity shock, the optimal amount of cash is still a decreasing function of the market liquidity λ. However, when λ approaches zero, the market liquidity is at its poorest and the fund provisions an optimal cash value approaching $\overline{\theta}$. Still, whatever the value of $\overline{\theta}$ and λ, the optimal cash value is also increasing in π. Looking at the four panels of Figure 3, we see that whatever the value of $\overline{\theta}$ the fund's expected value is increasing in the market liquidity λ and decreasing in the funding liquidity shock probability π, consistently with the intuition. Indeed, when the market liquidity increases, the fund reduces its optimal cash holding but earns an additional value from the average rebate being lower on the secondary market. The latter effect therefore dominates the former.

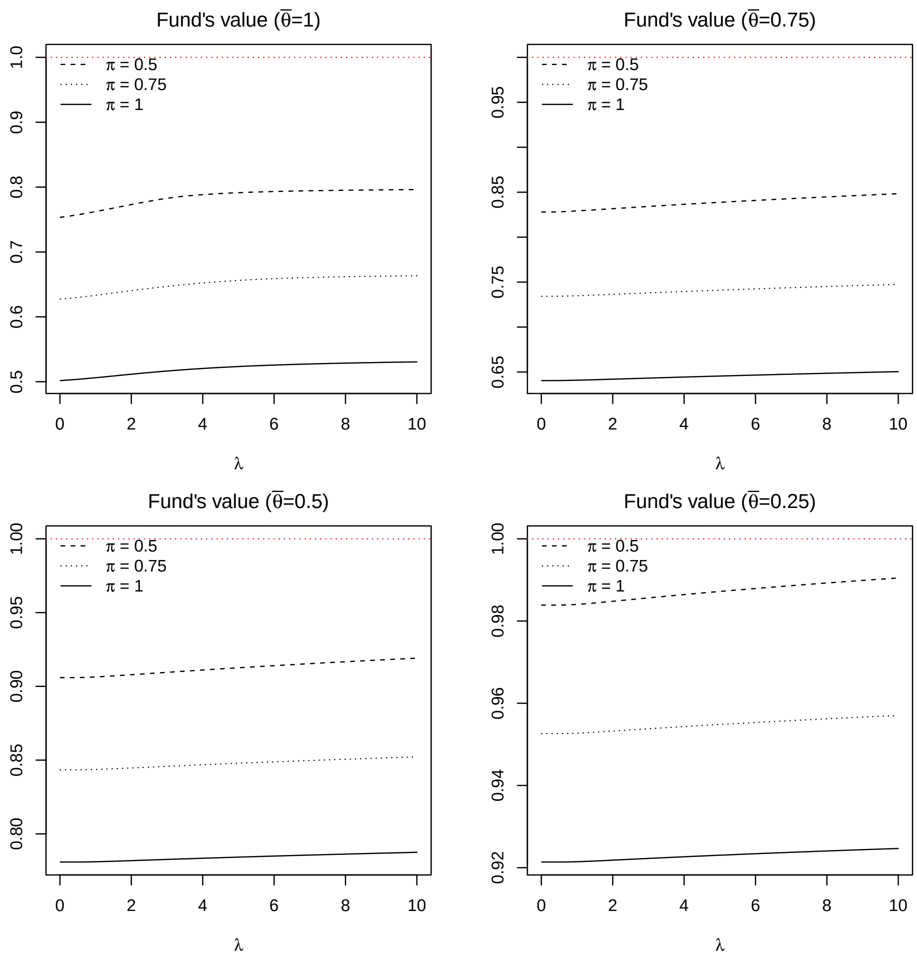
Figure 2: Optimal cash δ^* and fund's default probability



Notes: We take $\rho_1 = \rho_2 = 3\%$ in the calibration.

Last, looking at the right column of Figure 2, we still observe the bell-shaped pattern of the optimal default probabilities, but they are no longer superimposed for different values of π: the higher π, the lower the default probability. Again, the increase in the funding liquidity shock π has two effects. First, the fact that higher outflows are expected increases the default probability of the fund. However, the fund reacts in increasing its cash holding $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ which decreases the fund's default probability (see left column of Figure 2). The latter effect hence dominates the former.

Figure 3: Expected fund's value when cash is at the optimal $\delta^*(\overline{\theta}, \pi, \lambda, \rho_1, \rho_2)$



Notes: We take $\rho_1 = \rho_2 = 3\%$ in the calibration.

Proposition 3.5 *The optimal cash amount $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ is increasing in $\overline{\theta}$ when the market liquidity λ is low, and decreasing when the market liquidity λ is high. In contrast, the optimal default probability is always increasing in $\overline{\theta}$.*

Again, without being able to formally show these results, we refer to the computations presented on Figure 2. The effect is easily understood for the default probability since an increase in $\overline{\theta}$ increases the maximal size of the funding liquidity shock. Even though the optimal cash holding can increase for certain values of λ, it is not enough to compensate the increase in the risk and the default probability increases. For the optimal cash holding however, the sign of the effect depends on the value λ. For λ approaching 0, the liquidity is infinitely poor and the optimal cash holding approaches $\overline{\theta}$. Therefore, the variations of $\overline{\theta}$ and $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$ are nearly one-to-one (see left column of Figure 2). On the other hand, when λ is high enough, the fund knows that it can liquidate its assets at a reasonable average price. Instead of mitigating its default probability, when $\overline{\theta}$ increases, it also increases its holding of the illiquid asset such that its expected value does not decrease too much. This is particularly blatant on Figure 2 between $\overline{\theta} = 0.75$ and $\overline{\theta} = 1$: the slope of the optimal cash amount is steeper in the second case (top left graph), for any value of π.

4 Empirical Application

4.1 The data

The Lipper TASS database consists of monthly returns, Asset Under Management (AUM) and other HF characteristics for individual funds from May 1973 to October 2015.[4] The database categorizes HF into "Live" and "Graveyard" funds. We apply a series of filters to the data. First, we select only funds with Net Asset Value (NAV) written in USD, with monthly reporting frequency. This avoids double counting, since the same fund can have shares written in USD and Euro for example. We obtain 2353 funds in the "Live" base, and 8826 liquidated funds. Second, we only consider HF reporting their Asset Under Management on a regular basis. This information is essential to compute the time series of inflows and outflows. Third, to keep the interpretation in terms of individual funds, we eliminate the funds of funds and, for funds with multiple share classes, we eliminate duplicate share classes from the sample. Finally, we select the nine management styles with a sufficiently large size. These are Long/Short Equity Hedge (LSE), Event Driven (ED), Managed Futures (MF), Equity Market Neutral (EMN), Fixed Income Arbitrage (FI), Global Macro (GM), Emerging Markets (EM), Multi Strategy (MS), and Convertible Arbitrage (CONV). After applying all these filters, we get 1073 funds in the "Live" database and 5087 liquidated funds. The distribution by style of live and liquidated funds in the database is reported in Table 1.

The largest management style in the database of live and liquidated funds is Long/Short Equity Hedge (about 40%), followed by Managed Futures, Multi-Strategy and Event Driven (each about 10%). In the following, we only consider in the following the Long/Short Equity Hedge strategy. In addition, to obtain more reliable results, we start the sample in 2000 only where the number of funds is sufficient.

[4] Tremont Advisory Shareholders Services. Further information about this database is provided on the website http://www.lipperweb.com/products/LipperHedgeFundDatabase.aspx.

Table 1: The database.

	Live funds		Liquidated funds		Total	
	(#)	(%)	(#)	(%)	(#)	(%)
CONV	23	2.14	195	3.83	218	3.54
EM	153	14.26	495	9.73	648	10.52
EMN	41	3.82	362	7.12	403	6.54
ED	101	9.41	511	10.05	612	9.94
FI	35	3.26	208	4.09	243	3.94
GM	69	6.43	401	7.82	470	7.63
LSE	407	37.93	1914	37.63	2321	37.68
MF	134	12.49	573	11.26	707	11.48
MS	110	10.25	428	8.41	538	8.73
Total	**1073**	**100**	**5087**	**100**	**6160**	**100**

Notes: Codes are as follows: CONV: *convertible arbitrage*, EM: *emerging markets*, EMN: *equity market neutral*, ED: *event driven*, FI: *fixed income arbitrage*, GM: GLOBAL MACRO, LSE: *long/short equity hedge*, MF: *managed futures*, MS: *multi-strategy*. The table provides the distribution of iive funds on October 2015, and funds liquidated prior to October 2015, across the nine management styles.

4.2 Summary statistics

Figure 4, panels (a) and (b) display the variation of the subpopulation sizes and the liquidation rates overtime for the LSE management styles.

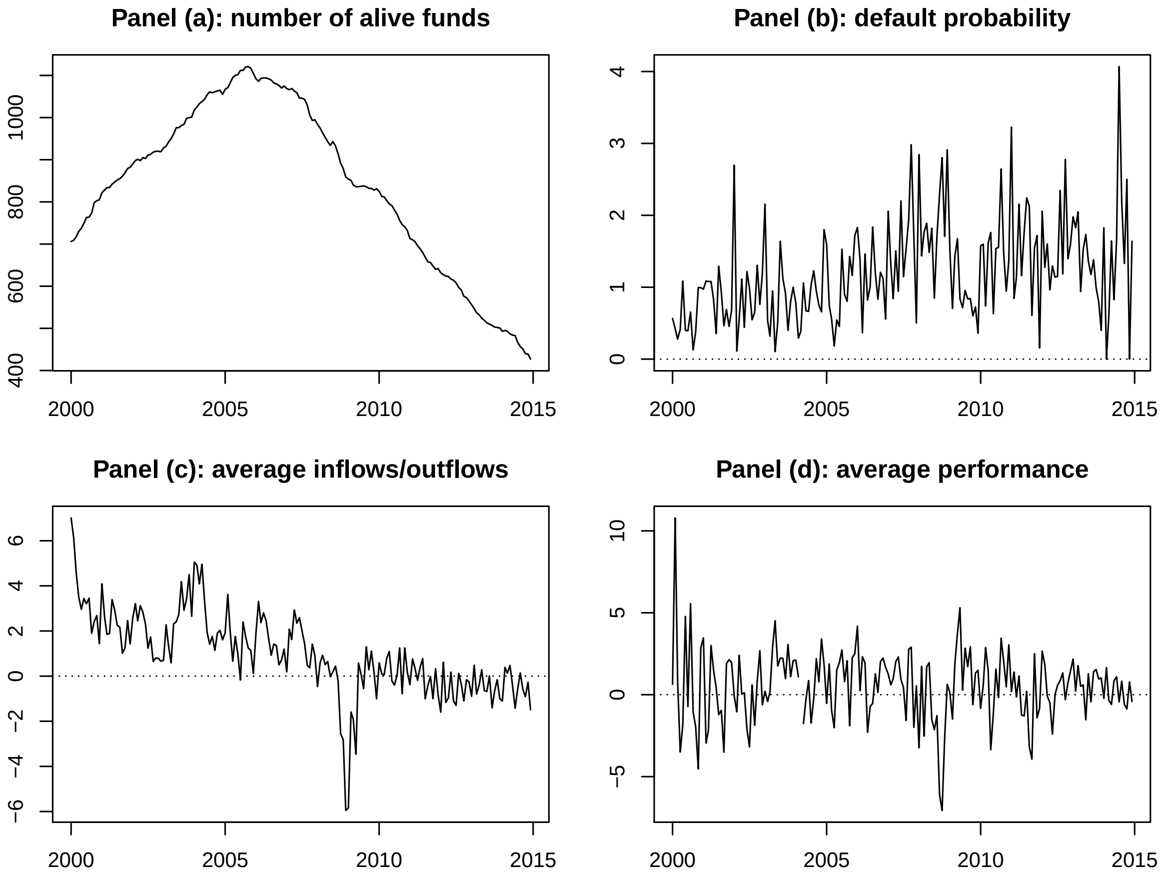
In Figure 4, panel(c), we observe the HF market growth between January 2000 and December 2014, and the sharp decrease due to the 2008 financial crisis.

4.3 Estimation method

In the model, there are 5 unknown quantities that are needed to obtain the optimal cash quantity $\delta^*(\overline{\theta}, \lambda, \pi, \rho_1, \rho_2)$. In this section, we design a simple calibration technique to obtain these values from the data. We denote the time index by $t \in \{1, \dots, \tau\}$ and the management style index by $i \in \{1, \dots, N\}$, where τ is the total number of periods in the sample and N is the total number of management styles.

First, for simplification, we assume that $\rho_{i,1} = \rho_{i,2} = \overline{\rho}_i$. We take this quantity to be fixed over time and calibrate it to the average return of the funds experiencing outflows at each period. Denote by $r_{j,t}^{(i)}$ and $f_{j,t}^{(i)}$ respectively the performance and the flow as a

Figure 4: Descriptive statistics: LSE management stye



Notes: The sample goes from January 2000 to December 2014.

percentage of the net asset value (inflows: $f_{j,t}^{(i)} > 0$) of fund j in management style i between $t-1$ and t, we have:

$$\overline{\rho}_i = \frac{1}{\tau}\sum_{t=1}^{\tau}\frac{1}{n_{i,t}}\sum_{j=1}^{n_{i,t}} r_{j,t}^{(i)}\mathbb{1}\{f_{j,t}^{(i)} < 0\}\,, \tag{10}$$

where $n_{i,t}$ is the number of alive funds in management style i at period t. The value $\pi_{i,t}$ is calibrated by taking the ratio of the number of funds who experience outflows at period t divided by the total number of funds for each management style. Formally:

$$\pi_{i,t} = \frac{1}{n_{i,t}}\sum_{j=1}^{n_{i,t}}\mathbb{1}\{f_{j,t}^{(i)} < 0\}\,, \tag{11}$$

The remaining parameters, $\overline{\theta}_{i,t}$, $\lambda_{i,t}$ and $\delta_{i,t}^*$ are obtained by minimizing a cost function in the spirit of the method of moments. We first match $\overline{\theta}_{i,t}$ with the maximum outflow observed across funds of management style i at time t. Since the maximum realized outflow may be an imperfect measure of the maximum possible outflow, we authorize $\overline{\theta}_{i,t}$ to be bigger than the observed data. This gives us a first cost function as:

$$m_1(\overline{\theta}_{i,t}) = \left(\overline{\theta}_{i,t} - \max_{j\in\{1,\ldots,n_{i,t}\}}(-f^{(i)}_{j,t})\right)^2 \quad \text{with} \quad \overline{\theta}_{i,t} \geqslant \max_{j\in\{1,\ldots,n_{i,t}\}}(-f^{(i)}_{j,t})\,. \tag{12}$$

Second, remember that given values of $(\overline{\theta}_{i,t}, \lambda_{i,t}, \pi_{i,t}, \overline{\rho}_i)$, it is possible to calculate the expected fund's value of Equation (8) hence to calculate the optimal cash quantity $\delta^*(\overline{\theta}_{i,t}, \lambda_{i,t}, \pi_{i,t}, \overline{\rho}_i)$. As emphasized previously, the maximization of expected value can be obtained numerically when $\overline{\theta}_{i,t} \neq 1$. Last, the average market liquidity $\lambda_{i,t}$ is matched with the data by considering the model-implied and the observed default probabilities. Given a value $\overline{\theta}_{i,t}$ and a $\delta^*_{i,t}$, the model-implied default probability is obtained with Equation (7). The empirical counterpart is simply the number of funds j of management style i disappearing between $t-1$ and t, denoted by $d^{(i)}_{j,t} = 1$, divided by the number of alive funds in the same management style in $t-1$. We thus obtain a second cost function which writes:[5]

$$m_2(\overline{\theta}_{i,t}, \lambda_{i,t}) = \left(\frac{\pi_{i,t}\left[\overline{\theta}_{i,t} - \delta^*(\overline{\theta}_{i,t}, \lambda_{i,t}, \pi_{i,t}, \overline{\rho}_i)\right]^{\lambda_{i,t}+1}}{\overline{\theta}_{i,t}(\lambda_{i,t}+1)\left[1 - \delta^*(\overline{\theta}_{i,t}, \lambda_{i,t}, \pi_{i,t}, \overline{\rho}_i)\right]^{\lambda_{i,t}}} \exp\left(-\lambda_{i,t}\overline{\rho}_i\right) - \frac{1}{n_{i,t-1}}\sum_{j=1}^{n_{i,t-1}} d^{(i)}_{j,t}\right)^2 . \tag{13}$$

For each period t and management style i, we obtain the following total cost function:

$$\mathcal{C}(\overline{\theta}_{i,t}, \lambda_{i,t}) = \alpha\, m_1(\overline{\theta}_{i,t}) + (1-\alpha) m_2(\overline{\theta}_{i,t}, \lambda_{i,t}) \quad \text{s.t.} \quad \begin{cases} \overline{\theta}_{i,t} \geqslant \max\limits_{j\in\{1,\ldots,n_{i,t}\}}(-f^{(i)}_{j,t}) \\ \lambda_{i,t} > 0 \end{cases} . \tag{14}$$

The weight α will typically be close to zero. Indeed, the cost function $m_2(\overline{\theta}_{i,t}, \lambda_{i,t})$ needs to be exactly equal to zero at the optimum whereas $\overline{\theta}_{i,t}$ can be greater than its lower bound. For each management style i and each period t, we minimize the total cost function $\mathcal{C}(\overline{\theta}_{i,t}, \lambda_{i,t})$ in both arguments imposing the constraints. This technique allows us to obtain time series for the secondary market liquidity $\lambda_{i,t}$, for the cash holdings $\delta^*_{i,t}$ and for the expected fund's value $F^*_{i,t}$ for each management style.

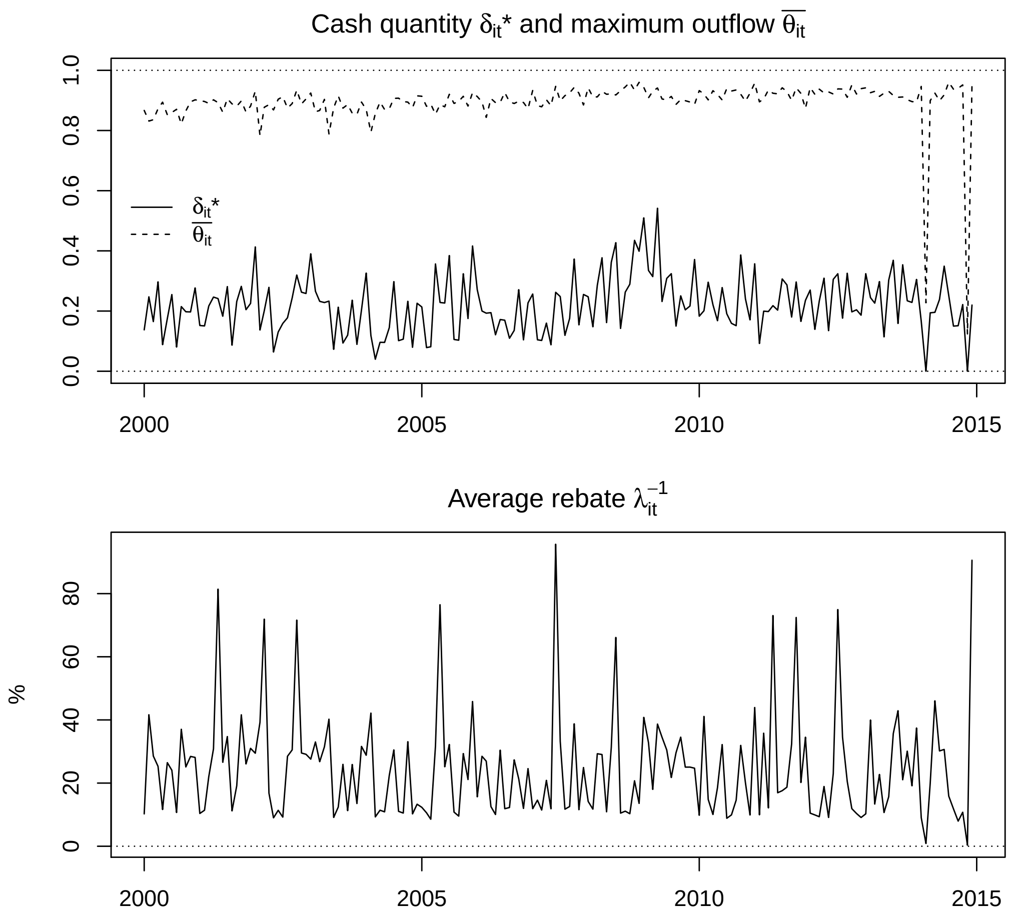
4.4 Estimation results

The series estimates for $\delta^*(\overline{\theta}_{\text{LSE},t}, \lambda_{\text{LSE},t}, \pi_{\text{LSE},t}, \overline{\rho}_{\text{LSE}})$ and $\overline{\theta}_{\text{LSE},t}$ are presented on the top graph of Figure 5, whereas the bottom graph presents the average rebate $\lambda^{-1}_{\text{LSE},t}$ for the

[5] For the sake of simplicity, we only kept $\overline{\theta}_{i,t}$ and $\lambda_{i,t}$ as the arguments of this second cost function even though $\pi_{i,t}$ and $\overline{\rho}_i$ intervene in the model-implied default probability formula.

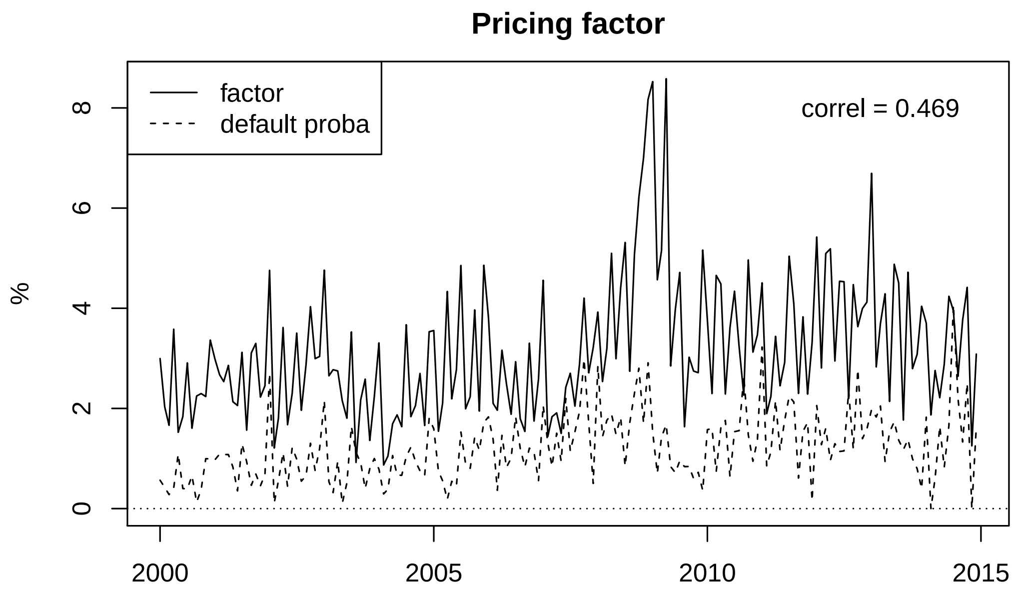
Long/Short Equity management style. On Figure 6, we present the corresponding pricing factor along with the default probability on the same graph. The correlation between the two series is about 0.5, which indicates that our pricing factor brings a somehow not disconnected but different information than the default probability.

Figure 5: Liquidity estimates: LSE management style



To be completed

Figure 6: Pricing factor: LSE management style



5 Conclusion

To be completed

A Appendix

Throughout the proofs gathered in the Appendices, we indifferently use the following notations:

- ℓ is the funding liquidity event: $\theta > 0$,
- ℓ_s is the small funding liquidity event: $\theta < \delta$,
- ℓ_b is the big funding liquidity event: $\theta > \delta$.

Hence, $\{\ell, \ell_s\}$ is the joint event of having a non-zero funding liquidity shock and that the shock is small. Note that ℓ_s and ℓ_b are mutually exclusive.

A.1 Default probability

A.1.1 Proof of Lemma 2.1

For the sake of notational simplicity, we first define $\widetilde{\theta} = \theta - \delta$. Conditionally on the combination of a liquidity shock happening and that the liquidation of the risky asset must be performed, we have:

$$\widetilde{\theta} | \theta > \delta \sim \mathcal{U}(0, \overline{\theta} - \delta)\,,$$

that is the conditional pdf of $\widetilde{\theta}$ is given by $f_{\widetilde{\theta}}(x) = \dfrac{\mathbb{1}_{\{x \in [0, \overline{\theta} - \delta]\}}}{\overline{\theta} - \delta}$. Remember that the conditional pdf of T is given by $f_T(x) = \mathbb{1}_{\{x \geqslant 0\}} \lambda \exp(-\lambda x)$ and that $\gamma_{\theta,T}$ is given by (see Equation (4)):

$$\gamma_{\theta,T} = \frac{\widetilde{\theta} \exp(T)}{(1-\delta)\exp(\rho_1)}\,.$$

Let us consider the invertible function $g(\widetilde{\theta}, T)$ defined by:

$$g\begin{pmatrix} \widetilde{\theta} \\ T \end{pmatrix} = \begin{pmatrix} \dfrac{\widetilde{\theta} \exp(T)}{(1-\delta)\exp(\rho_1)} \\ T \end{pmatrix} \Longleftrightarrow g^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x(1-\delta)\exp(\rho_1)}{\exp(y)} \\ y \end{pmatrix}.$$

The Jacobian matrix of $g^{-1}(x, y)$ is given by:

$$J_{g^{-1}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{(1-\delta)\exp(\rho_1)}{\exp(y)} & -x(1-\delta)\exp(\rho_1 - y) \\ 0 & 1 \end{pmatrix},$$

which determinant is simply given by: $(1-\delta)\exp(\rho_1 - y)$. The joint distribution of $(X,Y)' = g(\widetilde{\theta}, T)$ can be expressed via its pdf given by:

$$
\begin{aligned}
f_{X,Y}(x,y) &= (1-\delta)\exp(\rho_1 - y) f_{\widetilde{\theta},T}[g^{-1}(x,y)] \\
&= (1-\delta)\exp(\rho_1 - y) f_{\widetilde{\theta}}\left(\frac{x(1-\delta)\exp(\rho_1)}{\exp(y)}\right) f_T(y) \\
&= (1-\delta)\exp(\rho_1 - y)\left[\mathbb{1}_{\{y\geqslant 0\}}\lambda\exp(-\lambda y)\right]\left[\frac{\mathbb{1}_{\left\{x(1-\delta)\exp(\rho_1-y)\in[0,\overline{\theta}-\delta]\right\}}}{\overline{\theta}-\delta}\right] \\
&= \frac{\lambda(1-\delta)}{\overline{\theta}-\delta}\exp\left[\rho_1 - (\lambda+1)y\right]\mathbb{1}_{\{y\geqslant 0\}}\mathbb{1}_{\left\{x(1-\delta)\exp(\rho_1-y)\in[0,\overline{\theta}-\delta]\right\}}.
\end{aligned}
$$

To obtain the marginal distribution of $\gamma_{\theta,T}$, we integrate the previous distribution with respect to y.

$$
\begin{aligned}
f_X(x) &= \frac{\lambda(1-\delta)}{\overline{\theta}-\delta}\exp(\rho_1)\int_{\mathbb{R}}\exp\left[-(\lambda+1)y\right]\mathbb{1}_{\{y\geqslant 0\}}\mathbb{1}_{\left\{\exp(-y)\in\left[0,\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right]\right\}}\,\mathrm{d}y \\
&= \frac{\lambda(1-\delta)}{\overline{\theta}-\delta}\exp(\rho_1)\int_0^{+\infty}\exp\left[-(\lambda+1)y\right]\mathbb{1}_{\left\{y\geqslant -\log\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)\right\}}\,\mathrm{d}y \qquad (15)
\end{aligned}
$$

To know whether the indicator function plays a role in the integral, we look for the sign of its argument.

$$
\begin{aligned}
-\log\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)\geqslant 0 &\iff \frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\leqslant 1 \\
&\iff x\geqslant\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}. \qquad (16)
\end{aligned}
$$

We thus have two parts in the integral. When the condition stated by Equation (16) is verified, the integrated domain is the entire $\mathbb{R}_+$. Otherwise, the argument of the indicator represents a non-zero lower bound to the integrated domain. We can rewrite Equation (15) as follow:

$$
\begin{aligned}
f_X(x) &= \frac{\lambda(1-\delta)}{\overline{\theta}-\delta}\exp(\rho_1)\int_{\mathbb{1}_{\left\{x\geqslant\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}}\times\left[-\log\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)\right]}^{+\infty}\exp\left[-(\lambda+1)y\right]\,\mathrm{d}y \\
&= \frac{\lambda(1-\delta)}{\overline{\theta}-\delta}\exp(\rho_1)\left[\frac{-\exp\left[-(\lambda+1)y\right]}{\lambda+1}\right]_{\mathbb{1}_{\left\{x\geqslant\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}}\times\left[-\log\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)\right]}^{+\infty} \\
&= \frac{\lambda(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)}\left[\mathbb{1}_{\left\{x<\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}}+\mathbb{1}_{\left\{x\geqslant\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}}\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)^{\lambda+1}\right] \qquad (17)
\end{aligned}
$$

Even though this pdf is complicated, the conditional default probability given that the risky asset has to be liquidated can be derived in closed-form very easily. We first note that:

$$\begin{aligned}\overline{\theta} - \delta - (1-\delta)\exp(\rho_1) &\leqslant 1 - \delta - (1-\delta)\exp(\rho_1)\\ &\leqslant (1-\delta)(1-\exp(\rho_1))\\ &\leqslant 0\,,\end{aligned}$$

since $\rho_1 > 0$ by assumption. Therefore we have:

$$\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)} \leqslant 1\,.$$

We obtain the following simplification for the conditional probability of default:

$$\begin{aligned}\mathbb{P}(\gamma_{\theta,T} > 1|\ell,\ell_b) &= \int_1^{+\infty} \frac{\lambda(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)} \Bigg[\mathbb{1}_{\left\{x < \frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}} \\ &\quad + \mathbb{1}_{\left\{x \geqslant \frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}} \left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)^{\lambda+1}\Bigg]\mathrm{d}x \\ &= \frac{\lambda(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)} \int_1^{+\infty} \left[\mathbb{1}_{\left\{x \geqslant \frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}} \left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)^{\lambda+1}\right]\mathrm{d}x \\ &= \frac{\lambda(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)} \left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda+1} \int_1^{+\infty} \frac{1}{x^{\lambda+1}}\,\mathrm{d}x \\ &= \frac{\lambda}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda} \left[\frac{x^{-\lambda}}{-\lambda}\right]_1^{+\infty} \quad \text{whenever} \quad \lambda \neq 0 \\ &= \frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\end{aligned}$$

A.1.2 Proof of Proposition 2.1

Using Lemma 2.1, we obtain the total probability of default as:

$$\begin{aligned}\mathbb{P}(\gamma_{\theta,T} > 1) &= \pi\,\mathbb{P}(\gamma_{\theta,T} > 1|\ell) \\ &= \pi\,\mathbb{P}(\ell_b|\ell)\mathbb{P}(\gamma_{\theta,T} > 1|\ell,\ell_b) \\ &= \pi \times \frac{\overline{\theta}-\delta}{\overline{\theta}} \times \frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}.\end{aligned}$$

A.2 Proof of Corollary 2.1.1

$$
\begin{aligned}
\frac{\partial \mathbb{P}(\gamma_{\theta,T}>1)}{\partial \lambda} &= \pi \frac{\overline{\theta}-\delta}{\overline{\theta}} \times \frac{\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\left[(\lambda+1)\log\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)-1\right]}{(\lambda+1)^2} \\
&\leqslant 0 \quad \text{since} \quad \frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)} \in [0,1] \\
\frac{\partial \mathbb{P}(\gamma_{\theta,T}>1)}{\partial \overline{\theta}} &= \frac{\pi\exp(-\lambda\rho_1)}{(\lambda+1)(1-\delta)^{\lambda}} \times \frac{\overline{\theta}(\lambda+1)(\overline{\theta}-\delta)^{\lambda}-(\overline{\theta}-\delta)^{\lambda+1}}{\overline{\theta}^2} \\
&= \frac{\pi\exp(-\lambda\rho_1)}{(\lambda+1)(1-\delta)^{\lambda}} \times \frac{(\overline{\theta}-\delta)^{\lambda}}{\overline{\theta}^2}\left(\overline{\theta}\lambda+\delta\right) \geqslant 0 \\
\frac{\partial \mathbb{P}(\gamma_{\theta,T}>1)}{\partial \delta} &= \frac{\pi\exp(-\lambda\rho_1)}{(\lambda+1)\overline{\theta}} \times \frac{-(\lambda+1)(\overline{\theta}-\delta)^{\lambda}(1-\delta)^{\lambda}+\lambda(1-\delta)^{\lambda-1}(\overline{\theta}-\delta)^{\lambda+1}}{(1-\delta)^{2\lambda}} \\
&= \frac{\pi\exp(-\lambda\rho_1)}{(\lambda+1)\overline{\theta}} \frac{(1-\delta)^{\lambda-1}(\overline{\theta}-\delta)^{\lambda}}{(1-\delta)^{2\lambda}}\left[\lambda(\overline{\theta}-1)-(1-\delta)\right] < 0\,.
\end{aligned}
$$

A.3 Calculating expected fund's portfolio value

First, since ρ_1 and ρ_2 are known ex-ante, the expected fund's value $\mathbb{E}(F_{n\ell})$ are exactly equal to $F_{n\ell}$.

$$\mathbb{E}(F_{n\ell}) = (1-\delta)\exp(\rho_1+\rho_2)+\delta$$

Second, we can use the simple properties of the uniform distribution to calculate the second row of Equation (5). Note that the amount of cash δ chosen by the HF is always lower than the maximum liquidity shock $\overline{\theta}$ since the HF would have the same zero default probability and a higher return choosing $\delta = \overline{\theta}$.

$$
\begin{aligned}
\mathbb{P}(0<\theta\leqslant\delta)\,\mathbb{E}(F_{\ell_s}|\theta\in(0,\delta]) &= \frac{\delta}{\overline{\theta}} \times \left(F_{n\ell}-\mathbb{E}(\theta|\theta\in(0,\delta])\right) \\
&= \frac{\delta}{\overline{\theta}}\left(F_{n\ell}-\frac{\delta}{2}\right) \quad \text{since} \quad \delta<\overline{\theta} \\
&= \frac{\delta}{\overline{\theta}}\left((1-\delta)\exp(\rho_1+\rho_2)+\frac{\delta}{2}\right)
\end{aligned}
$$

Now, in order to express the expected value of expression (5), we now need to compute the expected fund's value in the case when there is a big liquidity shock (events ℓ and ℓ_b). We have:

$$\mathbb{E}(F_{\ell_b}|\ell,\theta>\delta,\gamma_{\theta,T}<1) = (1-\delta)\exp(\rho_1+\rho_2)\left[1-\mathbb{E}(\gamma_{\theta,T}|\ell,\theta>\delta,\gamma_{\theta,T}<1)\right]\,.$$

Note that the condition $\theta > \delta$ is exactly equivalent to the condition $\gamma_{\theta,T} > 0$. We therefore only need to compute $\mathbb{E}(\gamma_{\theta,T}|\ell,\gamma_{\theta,T}\in(0,1])$. This expectation is given by:

$$\mathbb{E}(\gamma_{\theta,T}|\gamma_{\theta,T}\in(0,1]) = \int x f_X(x) \mathbb{1}_{\{x\in(0,1]\}}\,\mathrm{d}x\,,$$

where $f_X(x)$ is the conditional pdf of $\gamma_{\theta,T}$ given by Equation (17). This expectation can be obtained as follow (whenever $\lambda\neq 1$):

$$\begin{aligned}
&\int x f_X(x)\mathbb{1}_{\{x\in(0,1]\}}\,\mathrm{d}x = \int_0^1 x f_X(x)\,\mathrm{d}x \\
=\;& \int_0^1 \frac{\lambda x(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)}\left[\mathbb{1}_{\left\{x<\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}} + \mathbb{1}_{\left\{x\geqslant\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right\}}\left(\frac{\overline{\theta}-\delta}{x(1-\delta)\exp(\rho_1)}\right)^{\lambda+1}\right]\mathrm{d}x \\
=\;& \int_0^{\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}} \frac{\lambda x(1-\delta)\exp(\rho_1)}{(\lambda+1)(\overline{\theta}-\delta)}\,\mathrm{d}x + \int_{\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}}^1 \frac{\lambda}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda} x^{-\lambda}\,\mathrm{d}x \\
=\;& \frac{\lambda(1-\delta)\exp(\rho_1)}{2(\lambda+1)(\overline{\theta}-\delta)}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^2 + \frac{\lambda}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\left[\frac{x^{-\lambda+1}}{-\lambda+1}\right]_{\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}}^1 \\
=\;& \frac{\lambda}{2(\lambda+1)}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right) + \frac{\lambda}{1-\lambda^2}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\left[1-\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{-\lambda+1}\right] \\
=\;& \frac{\lambda}{\lambda+1}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\left[\frac{1}{2}+\frac{1}{1-\lambda}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda-1} - \frac{1}{1-\lambda}\right] \\
=\;& \frac{\lambda}{1-\lambda^2}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\left[\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda-1} - \frac{\lambda+1}{2}\right], \qquad (18)
\end{aligned}$$

and, if $\lambda=1$:

$$\begin{aligned}
&\int x f_X(x)\mathbb{1}_{\{x\in(0,1]\}}\,\mathrm{d}x \\
=\;& \frac{\overline{\theta}-\delta}{4(1-\delta)\exp(\rho_1)} + \frac{1}{2}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\left[\log(x)\right]_{\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}}^1 \\
=\;& \frac{\overline{\theta}-\delta}{4(1-\delta)\exp(\rho_1)} - \frac{1}{2}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\log\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right) \\
=\;& \frac{\overline{\theta}-\delta}{2(1-\delta)\exp(\rho_1)}\left[\frac{1}{2}-\log\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)\right] \qquad (19)
\end{aligned}$$

In the end, the expression for expected fund's value is given by:

$$\begin{aligned}
\mathbb{E}(F) &= (1-\pi)\left[\delta + (1-\delta)\exp(\rho_1+\rho_2)\right] \\
&+ \pi\frac{\delta}{\overline{\theta}}\left((1-\delta)\exp(\rho_1+\rho_2) + \frac{\delta}{2}\right) \\
&+ \pi\frac{\overline{\theta}-\delta}{\overline{\theta}}\left[1 - \frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda}\right](1-\delta)\exp(\rho_1+\rho_2) \times \\
&\left\{1 - \frac{\lambda}{1-\lambda^2}\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\left[\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)^{\lambda-1} - \frac{\lambda+1}{2}\right]\right\}, \qquad (20)
\end{aligned}$$

when $\lambda \neq 1$, otherwise:

$$\begin{aligned}
\mathbb{E}_{\lambda=1}(F) &= (1-\pi)\left[\delta + (1-\delta)\exp(\rho_1+\rho_2)\right] \\
&+ \pi\frac{\delta}{\overline{\theta}}\left((1-\delta)\exp(\rho_1+\rho_2) + \frac{\delta}{2}\right) \\
&+ \pi\frac{\overline{\theta}-\delta}{\overline{\theta}}\left[1 - \frac{\overline{\theta}-\delta}{2(1-\delta)\exp(\rho_1)}\right](1-\delta)\exp(\rho_1+\rho_2) \times \\
&\left\{1 - \frac{\overline{\theta}-\delta}{2(1-\delta)\exp(\rho_1)}\left[\frac{1}{2} - \log\left(\frac{\overline{\theta}-\delta}{(1-\delta)\exp(\rho_1)}\right)\right]\right\}. \qquad (21)
\end{aligned}$$

A.4 Solving the model when $\overline{\theta} = 1$

A.4.1 Optimal cash solution $\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$

When a maximum of 100% of assets under management can go away at period $t = 1$, the expression for the expected fund's value of Equation (8) can be rewritten as:

$$\begin{aligned}
\mathbb{E}(F) &= (1-\pi)\left[\delta + (1-\delta)\exp(\rho_1+\rho_2)\right] + \pi\delta\left((1-\delta)\exp(\rho_1+\rho_2) + \frac{\delta}{2}\right) \\
&+ \pi(1-\delta)\left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda+1}\right](1-\delta)\exp(\rho_1+\rho_2) \times \\
&\left\{1 - \frac{\lambda}{1-\lambda^2}\exp(-\rho_1)\left[\exp[-(\lambda-1)\rho_1] - \frac{\lambda+1}{2}\right]\right\} \\
&= (1-\pi)\exp(\rho_1+\rho_2) \\
&+ \delta\left[(1-\pi)(1-\exp(\rho_1+\rho_2)) + \pi\exp(\rho_1+\rho_2)\right] \\
&+ \delta^2\left[\frac{\pi}{2} - \pi\exp(\rho_1+\rho_2)\right] \\
&+ \pi\left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda+1}\right]\left\{1 - \frac{\lambda\exp(-\rho_1)}{1-\lambda^2}\left[\exp[-(\lambda-1)\rho_1] - \frac{\lambda+1}{2}\right]\right\}\exp(\rho_1+\rho_2)(1-\delta)^2 \\
&\equiv a\delta^2 + b\delta + c\,, \qquad (22)
\end{aligned}$$

where

$$
\begin{aligned}
a &= \left[\frac{\pi}{2} - \pi \exp(\rho_1 + \rho_2)\right] \\
&+ \pi \left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda + 1}\right] \left\{1 - \frac{\lambda \exp(-\rho_1)}{1 - \lambda^2} \left[\exp[-(\lambda - 1)\rho_1] - \frac{\lambda + 1}{2}\right]\right\} \exp(\rho_1 + \rho_2) \\
b &= [(1 - \pi)(1 - \exp(\rho_1 + \rho_2)) + \pi \exp(\rho_1 + \rho_2)] \\
&- 2\pi \left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda + 1}\right] \left\{1 - \frac{\lambda \exp(-\rho_1)}{1 - \lambda^2} \left[\exp[-(\lambda - 1)\rho_1] - \frac{\lambda + 1}{2}\right]\right\} \exp(\rho_1 + \rho_2) \\
c &= (1 - \pi) \exp(\rho_1 + \rho_2) \\
&+ \pi \left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda + 1}\right] \left\{1 - \frac{\lambda \exp(-\rho_1)}{1 - \lambda^2} \left[\exp[-(\lambda - 1)\rho_1] - \frac{\lambda + 1}{2}\right]\right\} \exp(\rho_1 + \rho_2)
\end{aligned}
$$

Equivalently, when $\lambda = 1$, the expected fund's value can be written as a quadratic function of δ.

$$
\begin{aligned}
\mathbb{E}_{\lambda=1}(F) &\equiv \widetilde{a}\delta^2 + \widetilde{b}\delta + \widetilde{c} \\
&= (1 - \pi) \exp(\rho_1 + \rho_2) \\
&\quad + \delta \left[(1 - \pi)(1 - \exp(\rho_1 + \rho_2)) + \pi \exp(\rho_1 + \rho_2)\right] \\
&\quad + \delta^2 \left[\frac{\pi}{2} - \pi \exp(\rho_1 + \rho_2)\right] \\
&+ \pi(1 - \delta)^2 \left[1 - \frac{\exp(-\rho_1)}{2}\right] \exp(\rho_1 + \rho_2) \left\{1 - \frac{\exp(-\rho_1)}{2} \left[\frac{1}{2} + \rho_1\right]\right\} .
\end{aligned}
$$

Thus it is readily seen that:

$$
\begin{aligned}
\widetilde{a} &= \left[\frac{\pi}{2} - \pi \exp(\rho_1 + \rho_2)\right] + \pi \left[1 - \frac{\exp(-\rho_1)}{2}\right] \exp(\rho_1 + \rho_2) \left\{1 - \frac{\exp(-\rho_1)}{2} \left[\frac{1}{2} + \rho_1\right]\right\} \\
\widetilde{b} &= [(1 - \pi)(1 - \exp(\rho_1 + \rho_2)) + \pi \exp(\rho_1 + \rho_2)] \\
&- 2\pi \left[1 - \frac{\exp(-\rho_1)}{2}\right] \exp(\rho_1 + \rho_2) \left\{1 - \frac{\exp(-\rho_1)}{2} \left[\frac{1}{2} + \rho_1\right]\right\} \\
\widetilde{c} &= (1 - \pi) \exp(\rho_1 + \rho_2) + \pi \left[1 - \frac{\exp(-\rho_1)}{2}\right] \exp(\rho_1 + \rho_2) \left\{1 - \frac{\exp(-\rho_1)}{2} \left[\frac{1}{2} + \rho_1\right]\right\}
\end{aligned}
$$

The program of the HF is to maximize its expected value under the constraint that $\delta \in [0, 1]$. We compute the derivative of this expectation using the notation A, B, and C respectively for $\{a, \widetilde{a}\}$, $\{b, \widetilde{b}\}$ and $\{c, \widetilde{c}\}$.

$$
\frac{\partial \mathbb{E}(F)}{\partial \delta} = 0 \Longleftrightarrow \delta^* = -\frac{B}{2A} \tag{23}
$$

From now on, we consider only the case where $\lambda \neq 1$ for simplicity. Let us denote by:

$$\begin{aligned} G(\lambda, \rho_1) &= \left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda+1}\right] \left\{1 - \frac{\lambda\exp(-\rho_1)}{1-\lambda^2}\left[\exp[-(\lambda-1)\rho_1] - \frac{\lambda+1}{2}\right]\right\} \\ &= \left[1 - \frac{\exp(-\lambda\rho_1)}{\lambda+1}\right]\left[1 - \frac{\lambda\exp(-\lambda\rho_1)}{1-\lambda^2} + \frac{\lambda\exp(-\rho_1)}{2(1-\lambda)}\right] \\ &= [1 - p_c(\lambda, \rho_1)]\left(1 + \frac{\lambda\exp(-\rho_1)}{2(1-\lambda)} - \frac{\lambda\exp(-\lambda\rho_1)}{1-\lambda^2}\right). \end{aligned}$$

We hence have:

$$\begin{aligned} a &= \frac{\pi}{2} - \pi\exp(\rho_1+\rho_2) + \pi\exp(\rho_1+\rho_2)G(\lambda,\rho_1) \\ &= \pi\exp(\rho_1+\rho_2)\left[\frac{\exp(-\rho_1-\rho_2)}{2} - 1 + G(\lambda,\rho_1)\right] \\ b &= (1-\pi)(1-\exp(\rho_1+\rho_2)) + \pi\exp(\rho_1+\rho_2) - 2\pi\exp(\rho_1+\rho_2)G(\lambda,\rho_1) \\ &= -2\pi\exp(\rho_1+\rho_2)\left[G(\lambda,\rho_1) - \frac{1}{2} - \left(\frac{1-\pi}{2\pi}\right)(\exp(-\rho_1-\rho_2)-1)\right] \end{aligned}$$

Denoting by:

$$\begin{aligned} H_1(\pi,\rho_1,\rho_2) &= \frac{1}{2} + \left(\frac{1}{\pi} - 1\right)\left(\frac{1-\exp(-\rho_1-\rho_2)}{2}\right) \\ H_2(\rho_1,\rho_2) &= \frac{\exp(-\rho_1-\rho_2)}{2} - 1\,, \end{aligned}$$

We have:

$$\begin{aligned} a &= \pi\exp(\rho_1+\rho_2)\left(G(\lambda,\rho_1) + H_2(\rho_1,\rho_2)\right) \\ b &= -2\pi\exp(\rho_1+\rho_2)\left(G(\lambda,\rho_1) + H_1(\pi,\rho_1,\rho_2)\right). \end{aligned}$$

Hence we easily obtain:

$$\delta^*(1,\lambda,\pi,\rho_1,\rho_2) = \frac{G(\lambda,\rho_1) + H_1(\pi,\rho_1,\rho_2)}{G(\lambda,\rho_1) + H_2(\rho_1,\rho_2)}\,.$$

A.4.2 Partial derivatives of $\delta^*(1,\lambda,\pi,\rho_1,\rho_2)$

Let us now derive δ^* with respect to its different components. The influence of λ is given by:

$$\text{sign}\left[\frac{\partial\delta^*(1,\lambda,\pi,\rho_1,\rho_2)}{\partial\lambda}\right] = \text{sign}\left[\frac{\partial G(\lambda,\rho_1)}{\partial\lambda}\left(H_2(\rho_1,\rho_2) - H_1(\pi,\rho_1,\rho_2)\right)\right].$$

Remember that the function $G(\lambda, \rho_1)$ is given by the product of the conditional survival probability $(1 - p_c(\lambda, \rho_1))$ and the conditional expectation of $1 - \gamma_{\theta,T}$ given a big liquidity shock **and** the fund's survival $(\gamma_{\theta,T} < 1)$. It is obvious that these two functions are positive and constrained between 0 and 1 by definition. Using the formula of Proposition 2.1, it is easy to see that $p_c(\lambda, \rho_1)$ is decreasing in λ, hence $1 - p_c(\lambda, \rho_1)$ is an increasing function of λ. Second, when $\overline{\theta} = 1$, the conditional expectation of $\gamma_{\theta,T}$ does not depend on δ (see the last row of Equation (8)). Therefore, given any value of δ chosen by the fund, it is clear that the conditional expectation of $\gamma_{\theta,T}$ is decreasing in λ. Indeed, the more liquid the secondary market is, the less the fund needs to sell its illiquid asset. Hence the second term in $G(\lambda, \rho_1)$ is increasing in λ. We conclude that $G(\lambda, \rho_1)$ is increasing in λ and $\partial G(\lambda, \rho_1)/\partial\lambda \geqslant 0$. We thus have:

$$\begin{aligned}
\text{sign}\left[\frac{\partial\delta^*(1,\lambda,\pi,\rho_1,\rho_2)}{\partial\lambda}\right] &= \text{sign}\left[H_2(\rho_1,\rho_2) - H_1(\pi,\rho_1,\rho_2)\right] \\
&= \frac{\exp(-\rho_1-\rho_2)}{2} - 1 + \left(1 - \frac{1}{\pi}\right)\left(\frac{1-\exp(-\rho_1-\rho_2)}{2}\right) + \frac{1}{2} \\
&= \frac{\exp(-\rho_1-\rho_2) - 1}{2\pi} < 0\,.
\end{aligned}$$

$\delta^*(1, \lambda, \pi, \rho_1, \rho_2)$ is therefore a decreasing function of λ.

The influence of π is given by:

$$\begin{aligned}
\text{sign}\left[\frac{\partial\delta^*(1,\lambda,\pi,\rho_1,\rho_2)}{\partial\pi}\right] &= \text{sign}\left[\frac{\partial H_1(\pi,\rho_1,\rho_2)}{\partial\pi}\left(G(\lambda,\rho_1) + H_2(\rho_1,\rho_2)\right)\right] \\
&= \text{sign}\left[\frac{\exp(-\rho_1-\rho_2)-1}{2\pi^2}\left(G(\lambda,\rho_1) + H_2(\rho_1,\rho_2)\right)\right] \\
&= -\text{sign}\left[G(\lambda,\rho_1) + H_2(\rho_1,\rho_2)\right]\,.
\end{aligned}$$

Since $H_2(\rho_1, \rho_2) - H_1(\pi, \rho_1, \rho_2) < 0$, we have:

$$\begin{aligned}
& G(\lambda,\rho_1) + H_1(\pi,\rho_1,\rho_2) > G(\lambda,\rho_1) + H_2(\rho_1,\rho_2) \\
\iff & \begin{cases} \frac{G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)} > 1 & \textbf{if} \quad G(\lambda,\rho_1) + H_2(\rho_1,\rho_2) > 0 \\ \frac{G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)} < 1 & \textbf{if} \quad G(\lambda,\rho_1) + H_2(\rho_1,\rho_2) < 0 \end{cases}
\end{aligned}$$

Since $\delta^* \in [0, 1]$, this implies that $G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)$ must be negative, and $G(\lambda, \rho_1) + H_1(\pi, \rho_1, \rho_2)$ must be negative as well. As a consequence, we obtain:

$$\frac{\partial\delta^*(1,\lambda,\pi,\rho_1,\rho_2)}{\partial\pi} < 0$$

A.4.3 Derivative of the expected fund's value

Since the expected fund's value is a quadratic function of δ, the maximum excess return is equal to:

$$R^* = c - \frac{b^2}{4a} \quad \text{whenever} \quad \frac{-b}{2a} \in (0,1)\,.$$

Using the same notations as before, it is readily seen that:

$$\begin{aligned} c &= (1-\pi)\exp(\rho_1+\rho_2) + \pi\exp(\rho_1+\rho_2)G(\lambda,\rho_1) \\ &= \exp(\rho_1+\rho_2)\left[1+\pi G(\lambda,\rho_1)-\pi\right]\,. \end{aligned}$$

Therefore:

$$\begin{aligned} F^* &= e^{\rho_1+\rho_2}\left[1+\pi G(\lambda,\rho_1)-\pi\right] - \left(\frac{\left[2\pi e^{\rho_1+\rho_2}(G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2))\right]^2}{4\pi e^{\rho_1+\rho_2}(G(\lambda,\rho_1)+H_2(\rho_1,\rho_2))}\right) \\ &= \pi e^{\rho_1+\rho_2}\left\{\frac{1}{\pi} + G(\lambda,\rho_1) - 1 - \frac{\left[G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)\right]^2}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)}\right\} \\ &= \pi e^{\rho_1+\rho_2}\left\{\frac{1}{\pi} - \frac{e^{-\rho_1-\rho_2}}{2} + \frac{\left[G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)\right]^2 - \left[G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)\right]^2}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)}\right\} \\ &= \pi e^{\rho_1+\rho_2}\left\{\frac{1}{\pi} - \frac{e^{-\rho_1-\rho_2}}{2} + \frac{\left[2G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)+H_2(\rho_1,\rho_2)\right]\left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]}{G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)}\right\}\,. \end{aligned}$$

The derivative of F^* with respect to λ is given by:

$$\begin{aligned} \frac{\partial F^*(\lambda,\pi,\rho_1,\rho_2)}{\partial\lambda} &= \pi e^{\rho_1+\rho_2}\left\{\frac{2\left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]\left[G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)\right]\frac{\partial G(\lambda,\rho_1)}{\partial\lambda}}{\left[G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)\right]^2}\right. \\ &\quad - \left.\frac{\frac{\partial G(\lambda,\rho_1)}{\partial\lambda}\left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]\left[2G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)+H_2(\rho_1,\rho_2)\right]}{\left[G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)\right]^2}\right\} \\ &\propto 2\left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]\left[G(\lambda,\rho_1)+H_2(\rho_1,\rho_2)\right] \\ &\quad -\left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]\left[2G(\lambda,\rho_1)+H_1(\pi,\rho_1,\rho_2)+H_2(\rho_1,\rho_2)\right] \\ &= \left[H_2(\rho_1,\rho_2)-H_1(\pi,\rho_1,\rho_2)\right]^2 \geqslant 0\,. \end{aligned}$$

Hence the expected returns are increasing in λ.

A.5 Default probabilities at the optimum cash amount when $\overline{\theta} = 1$

When $\overline{\theta} = 1$ and given the optimal quantity of cash, the total default probability is given by (see Equation (7)):

$$\begin{aligned}
\mathbb{P}(\gamma_{\theta,T} > 1) &= \pi(1 - \delta^*(1, \lambda, \pi, \rho_1, \rho_2)p_c(\lambda, \rho_1) \\
&= \pi \left(\frac{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2) - [G(\lambda, \rho_1) + H_1(\pi, \rho_1, \rho_2)]}{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)} \right) p_c(\lambda, \rho_1) \\
&= \pi \left(\frac{H_2(\rho_1, \rho_2) - H_1(\pi, \rho_1, \rho_2)}{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)} \right) p_c(\lambda, \rho_1),.
\end{aligned}$$

The influence of π is given by:

$$\text{sign}\frac{\partial \mathbb{P}(\gamma_{\theta,T} > 1)}{\partial \pi} = \text{sign}\frac{H_2(\rho_1, \rho_2) - H_1(\pi, \rho_1, \rho_2) - \pi \frac{\partial H_1(\pi, \rho_1, \rho_2)}{\partial \pi}}{G(\lambda, \rho_1) + H_2(\rho_1, \rho_2)}$$

The numerator is given by:

$$\frac{e^{-\rho_1 - \rho_2}}{2} - 1 + \left(1 - \frac{1}{\pi}\right) \frac{1 - e^{-\rho_1 - \rho_2}}{2} + \frac{1}{2} + \frac{1 - e^{-\rho_1 - \rho_2}}{2\pi} = 0\,.$$

As a consequence, π has no influence on the default probability of the fund when $\overline{\theta} = 1$.

Let us turn now to the influence of λ on the default probability. **DEMO**

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